

LIPSCHITZ CONDITIONS, TRIANGULAR RATIO METRIC, AND QUASICONFORMAL MAPS

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ABSTRACT. The triangular ratio metric is studied in subdomains of the complex plane and Euclidean n -space. Various inequalities are proven for it. The main results deal with the behavior of this metric under quasiconformal maps. We also study the smoothness of metric disks with small radii.

1. INTRODUCTION

A significant part of geometric function theory deals with the behavior of distances under well known classes of mappings such as Möbius transformations, bilipschitz maps or quasiconformal mappings. Thus measurement of distances in terms of metrics is a common tool in function theory and frequently hyperbolic metrics or metrics of hyperbolic type are used in addition to Euclidean or chordal distance. Many authors have contributed to this development in recent years. See for instance [H], [HIMPS], [KL], [PT]. A survey of these developments is given in [Vu2].

The triangular ratio metric is defined as follows for a domain $G \subsetneq \mathbb{R}^n$ and $x, y \in G$:

$$(1.1) \quad s_G(x, y) = \sup_{z \in \partial G} \frac{|x - y|}{|x - z| + |z - y|} \in [0, 1].$$

Clearly, the supremum in the definition (1.1) of s_G is attained at some point $z \in \partial G$, but finding this point is a nontrivial problem even for the case when G is the unit disk. P. Hästö [H, Theorem 6.1] proved that s_G satisfies the triangle inequality and developed theory for metrics more general than s_G and generalized the work of A. Barrlund [BA]. Very recently, the geometry of the balls of s_G for some special domains was studied in [HKLV]. Our goal here is to continue the study of this metric and to explore its behavior under Möbius transformations, quasiconformal and quasiregular mappings. We also give upper and lower bounds for this metric in terms of other metrics in several domains such as the unit ball, the upper half plane and $\mathbb{R}^n \setminus \{0\}$, the whole space \mathbb{R}^n punctured at the origin. Also some ideas for further work are pointed out.

The paper is divided into sections as follows. In Section 2 we give algorithms for numerically finding the value of $s_G(x, y)$, for instance, in the case of a domain bounded by a polygon. In Section 3 we develop the main ideas of

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this paper and relate the triangular ratio metric to other well-known metrics of geometric function theory such as the hyperbolic metric of the unit ball or half-space or to the distance ratio metric of a domain $G \subset \mathbb{R}^n$. In Section 5 apply these results and well-known distortion results of quasiconformal maps to study how the triangular ratio metric behaves under quasiconformal and quasiregular mappings. In Section 4 we study the smoothness of the boundaries of s -disks in a triangle and in a rectangle. We now proceed to formulate some of our main results.

Theorem 1.2. (1) Let $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$ be a K -quasiregular mapping. Then for $x, y \in \mathbb{H}^n$ we have

$$(1.3) \quad s_{\mathbb{H}^n}(f(x), f(y)) \leq \lambda_n^{1-\alpha} (s_{\mathbb{H}^n}(x, y))^\alpha, \quad \alpha = K^{1/(1-n)},$$

where $\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$, is the Grötzsch ring constant depending only on n ([Vu1, Lemma 7.22]).

(2) Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^n$ be a K -quasiregular mapping. Then for $x, y \in \mathbb{B}^n$ we have

$$(1.4) \quad s_{\mathbb{B}^n}(f(x), f(y)) \leq 2^\alpha \lambda_n^{1-\alpha} (s_{\mathbb{B}^n}(x, y))^\alpha, \quad \alpha = K^{1/(1-n)}.$$

Theorem 1.5. Let $G = \mathbb{R}^n \setminus \{0\}$, and $f : G \rightarrow G$ be a K -quasiconformal mapping with $f(\infty) = \infty$, and let z, w be two distinct points in G and $\alpha = K^{1/(1-n)}$. Then

$$(1.6) \quad s_{fG}(f(z), f(w)) \leq \frac{1}{P_5(n, K)} (s_G(z, w))^\alpha, \quad s_G(z, w) = \frac{|z - w|}{|z| + |w|},$$

where $P_5(n, K) \rightarrow 1, K \rightarrow 1$ is defined in Lemma 5.4.

Of particular interest is the special case $K = 1$ of Theorems 1.2 and 1.5. Clearly, Theorem 1.5 is sharp in this case and the same is true about Theorem 1.2 (1). The question about the best constant in Theorem 1.2 (2) deserves some attention for the case when $K = 1 = \alpha$. The constant on the right hand side is then 2.

For a detailed study of this constant we define a given point $a \in \mathbb{B}^n$ and a Möbius transformation T_a on \mathbb{B}^n onto \mathbb{B}^n with $T_a(a) = 0$, the constant

$$(1.7) \quad L(a) = \sup\{s_{\mathbb{B}^n}(T_a(x), T_a(y)) / s_{\mathbb{B}^n}(x, y) : x, y \in \mathbb{B}^n, x \neq y\}.$$

Theorem 1.8. $L(a) \geq 1 + |a|$.

Theorem 1.8 shows that for $K = 1$ the constant 2 in Theorem 1.2 (2) cannot be replaced by a smaller constant (independent of $|a|$).

Conjecture 1.9. Our numerical experiments for $n = 2$ suggest that $L(a) = 1 + |a|$.

In Theorem 3.35 we show that $L(a) \leq \frac{1+|a|}{1-|a|}$.

For a domain $G \subset \mathbb{R}^n, x, y \in G$, we define the j -metric by

$$j_G(x, y) = \log \left(1 + \frac{|x - y|}{\min\{d_G(x), d_G(y)\}} \right),$$

where $d_G(z) = d(z, \partial G)$. We will omit the subscript G if it is clear from context. This metric has found numerous applications in geometric function theory, see [HIMPS, Vu1]. We also define

$$p_G(x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + 4 d_G(x) d_G(y)}}.$$

We next formulate some of our comparison results between metrics.

Theorem 1.10. *Let G be a proper subdomain of \mathbb{R}^n . Then for all $x, y \in G$ we have*

$$p_G(x, y) \leq C j_G(x, y), \quad C = \max \left\{ \frac{2 + \mu}{4}, \frac{1}{\log(1 + \mu)} \right\}, \quad 0 < \mu < 1$$

and

$$s_G(x, y) \leq \frac{1}{\log 3} j_G(x, y),$$

where the constant $\frac{1}{\log 3} \approx 0.91$ is the best possible.

Theorem 1.11. (1) *Let $t \in (0, 1)$ and $m \in \{j, p, s\}$. There exists a constant $c_m = c_m(t) > 1$ such that for all $x, y \in \mathbb{B}^n$ with $|x|, |y| < t$ we have*

$$m_{\mathbb{B}^n}(x, y) \leq c_m m_{\mathbb{R}^n \setminus \{e_1\}}(x, y).$$

Moreover, $c(t) \rightarrow 1$ as $t \rightarrow 0$ and $c(t) \rightarrow \infty$ as $t \rightarrow 1$.

(2) *Let $G \subset \mathbb{R}^n$, $x \in G$, $t \in (0, 1)$ and $m \in \{j, p, s\}$. Then there exists a constant $c_m = c_m(t)$ such that for all $y, z \in G \setminus B(x, td_G(x))$ we have*

$$m_{G \setminus \{x\}}(y, z) \leq c_m m_G(y, z).$$

Moreover, the constant is best possible as $t \rightarrow 1$. This means that $c_j, c_p, c_s \rightarrow 2$ as $t \rightarrow 1$.

We also study the geometry of disks of the s -metric. We use the notation

$$B_{s_G}(x, r) = \{z \in G : s_G(x, z) < r\}$$

for the balls of the s -metric. First we show that disks of small enough radii have smooth boundaries and our main result here is Theorem 1.12.

Let us denote $T_{\frac{\pi}{6}, 2}$ the equilateral triangle with vertices $(0, 0)$, $(\sqrt{3}, 1)$, $(\sqrt{3}, -1)$, and $R_{a,b}$ the rectangle with vertex points (a, b) , $(a, -b)$, $(-a, b)$, $(-a, -b)$, where $a \geq b > 0$.

Theorem 1.12. (1) *Let $G = T_{\frac{\pi}{6}, 2}$, $x = (x_1, x_2) \in G$, $r > 0$. Then the metric ball $B_{s_G}(x, r)$ is smooth if and only if $r \leq r_0$ or $r \leq r_1$, where*

$$r_0 = \min \left\{ \frac{2|x_2|}{|x|}, \frac{|x_2| - \sqrt{3}x_1 + 2}{\sqrt{(x_1 - \sqrt{3})^2 + (1 - |x_2|)^2}} \right\}, \text{ and } r_1 = \frac{\sqrt{3}x_1 - 2 - |x_2|}{\sqrt{(x_1 - \sqrt{3})^2 + (1 - |x_2|)^2}}.$$

(2) *Let $G = R_{a,b}$, $x = (x_1, x_2) \in G$, $r > 0$. Then the metric ball $B_{s_G}(x, r)$ is smooth if and only if $r \leq r_2$ or $r \leq r_3$, where*

$$r_2 = \min \left\{ \frac{|x_2|}{b}, \frac{(a - |x_1|) - (b - |x_2|)}{\sqrt{(a - |x_1|)^2 + (b - |x_2|)^2}} \right\}, \text{ and } r_3 = \min \left\{ \frac{|x_1|}{a}, \frac{(b - |x_2|) - (a - |x_1|)}{\sqrt{(a - |x_1|)^2 + (b - |x_2|)^2}} \right\}.$$

2. ALGORITHMS FOR NUMERICAL COMPUTATION OF s_G

The hyperbolic metric $\rho_{\mathbb{H}^n}$ and $\rho_{\mathbb{B}^n}$ of the upper half plane $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ and of the unit ball $\mathbb{B}^n = \{z \in \mathbb{R}^n : |z| < 1\}$ can be defined as weighted metrics with the weight functions $w_{\mathbb{H}^n}(x) = 1/x_n$ and $w_{\mathbb{B}^n}(x) = 2/(1 - |x|^2)$, respectively. This definition as such is rather abstract and for applications concrete formulas are needed. By [B, p.35] we have

$$(2.1) \quad \cosh \rho_{\mathbb{H}^n}(x, y) = 1 + \frac{|x - y|^2}{2x_n y_n}$$

for all $x, y \in \mathbb{H}^n$, and by [B, p.40] we have

$$(2.2) \quad \sinh \frac{\rho_{\mathbb{B}^n}(x, y)}{2} = \frac{|x - y|}{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}}$$

and

$$(2.3) \quad \begin{aligned} \tanh \frac{\rho_{\mathbb{B}^n}(x, y)}{2} &= \frac{|x - y|}{\sqrt{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}} \\ &= \frac{|x - y|}{|x||x^* - y|}, \quad x^* = \frac{x}{|x|^2}, \end{aligned}$$

(2.4)

for all $x, y \in \mathbb{B}^n \setminus \{0\}$. As shown in [HKL, Theorem 4.2] we have

$$(2.5) \quad s_{\mathbb{H}^n}(x, y) = \tanh \frac{\rho_{\mathbb{H}^n}(x, y)}{2} = \frac{|x - y|}{|x - \bar{y}|},$$

for all $x, y \in \mathbb{H}^n$, where \bar{y} is the reflection of y with respect to $\partial\mathbb{H}^n$. See also (2.9) below. Unfortunately, there is no formula similar to (2.5) for the case of $s_{\mathbb{B}^n}$. Therefore inequalities for $s_{\mathbb{B}^n}$ are needed, see Section 3 below.

Explicit formulas for $s_G(x, y)$ are known only for a few particular cases. Our goal is to list several domains for which we have written algorithms in the MATLAB language. The definition of $s_G(x, y)$ readily shows that the supremum is attained and that a point $z \in \partial G$ with $s_G(x, y) = \frac{|x - y|}{|x - z| + |z - y|}$ is located on the maximal ellipse with foci x and y and contained in \overline{G} . The point z is called an extremal point. Finding this maximal ellipse is however a difficult task even for \mathbb{B}^2 . In the course of this research we have extensively made use of experiments using the algorithms in this section. In particular, Conjecture 1.9 is based on these algorithms.

Algorithm 2.6. $s_{\mathbb{B}^2}$

Let $x, y \in \mathbb{B}^2$ and $z \in \partial\mathbb{B}^2$ be such that

$$(2.7) \quad s_{\mathbb{B}^2}(x, y) = \frac{|x - y|}{|x - z| + |z - y|}.$$

The point z can be found by standard minimization algorithm on the smaller arc on $\partial\mathbb{B}^2$ between x and y .

Algorithm 2.8. $s_{\mathbb{H}^2}$

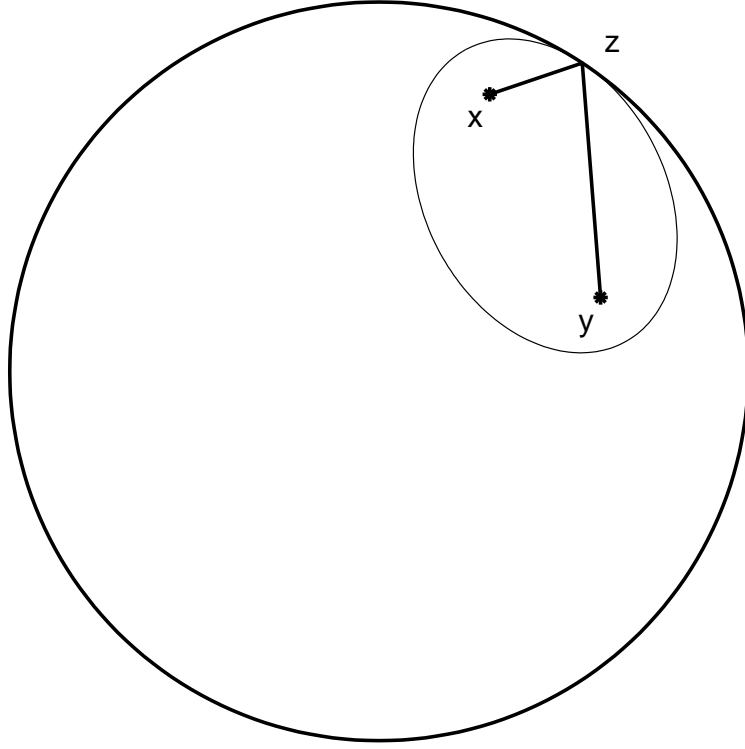


FIGURE 1. The maximal ellipse with foci x and y and contained in $\overline{\mathbb{B}^2}$.

Suppose that $x, y \in \mathbb{H}^2$ are two distinct points. An extremal point $z \in \partial\mathbb{H}^2 = \mathbb{R}$ for $s_{\mathbb{H}^2}(x, y)$ minimizes the sum

$$|x - z| + |z - y| = |x - z| + |z - \bar{y}|.$$

Therefore z is the unique point of intersection of the segment $[x, \bar{y}]$ with the real axis. In conclusion,

$$(2.9) \quad s_{\mathbb{H}^2}(x, y) = \frac{|x - y|}{|x - \bar{y}|}.$$

Algorithm 2.10. s_R , R is a rectangle

Given distinct x, y in a rectangle R , the extremal boundary point z as in (1.1) must be located on one of the four sides $T_j, j = 1, \dots, 4$ of R . If y_j is the reflection point of y with respect to side $T_j, j = 1, \dots, 4$, then $z_j = [x, y_j] \cap \partial R$ and

$$(2.11) \quad s_R(x, y) = \frac{|x - y|}{\min\{|x - y_j| : j = 1, 2, 3, 4\}}.$$

Algorithm 2.12. s_A , A is a sector

Let $\alpha \in (0, \pi)$ and $A = \{z \in \mathbb{C} : 0 < \arg z < \alpha\}$. Given $x, y \in A$, the extremal point $z \in \partial A$ for $s_A(x, y)$ has only two options: it is located either on the real axis $\{x \in \mathbb{R} : x \geq 0\}$ or on the ray $\{t \exp i\alpha : t > 0\}$. In the first case by (2.9)

$$s_A(x, y) = \frac{|x - y|}{|x - \bar{y}|},$$

whereas in the second case again by (2.9)

$$s_A(x, y) = \frac{|x - y|}{|x - y_2|},$$

where $y_2 = |y| \exp i(2\alpha - \arg y)$. In conclusion, in both cases

$$(2.13) \quad s_A(x, y) = \frac{|x - y|}{\min\{|x - \bar{y}|, |x - y_2|\}}.$$

This idea can be extended in a straightforward way to triangles and other convex polygons.

Algorithm 2.14. s_P , P polygon

Suppose that v_1, v_2, \dots, v_m are points in the plane such that the polygon with these points as vertices is a bounded Jordan domain. We consider two methods:

- Method I.

Based on exhaustive tabulation of function values and choosing the optimal point on ∂P . We parameterize ∂P using the polygonal curve length as a parameter, measured from v_1 via the points v_j . Then this real parameter varies on $[0, L]$ where

$$L = \sum_{j=1}^m |v_j - v_{j+1}|,$$

and we agree that $v_{m+1} = v_1$. The parametrization $z : [0, L] \rightarrow \partial P$ enables us to find all the competing points for the definition of $s_P(x, y)$. Then finding $s_P(x, y)$ becomes a 1-dimensional minimization problem, which can be solved by standard methods.

- Method II.

This method makes use of standard minimization algorithms for finding the point z . Minimization is carried out separately for each side in the same way as in the case of rectangle.

3. COMPARISON RESULTS FOR s_G

From the definition (1.1) of s_G it is clear that s_G has three important properties:

- (a) *monotonicity with respect to domain*, i.e. if $D_1, D_2 \subset \mathbb{R}^n$ are domains with $D_1 \subset D_2$ and $x, y \in D_1$, then $s_{D_1}(x, y) \geq s_{D_2}(x, y)$, and
- (b) *sensitivity to boundary variation*, i.e. if $D \subset \mathbb{R}^n$ is a domain and $x_0 \in D$, then the numerical values of $s_D(x, y)$ and $s_{D \setminus \{x_0\}}(x, y)$ are not comparable if x, y are very close to x_0 .
- (c) For fixed $x, y \in G$, one extremal boundary point $z \in \partial G$ determines the numerical value of $s_G(x, y)$.

In this section our goal is to find various inequalities for s_G in terms of expressions that are explicit. In particular, we hope to get rid of the infimum in (1.1), and hope to use expressions that have the above properties (a)–(c). Most of these expressions define metrics and we will show that these metrics are locally quantitatively equivalent.

For a domain $G \subset \mathbb{R}^n$, $x, y \in G$, we define *the visual angle metric* [KLVW] by

$$v_G(x, y) = \sup\{\angle(x, z, y) : z \in \partial G\}.$$

The metrics j_G , v_G and s_G have the aforementioned three properties (a)–(c) and $p_G \leq 1$, $v_G \leq \pi$ while j_G is unbounded. All of the expressions s_G, v_G, j_G, p_G are invariant under similarity transformations.

Remark 3.1. Because the inequality $p_{\mathbb{B}^2}(t, 0) + p_{\mathbb{B}^2}(0, -t) > p_{\mathbb{B}^2}(t, -t)$, fails for small t , we see that p_G is not a metric.

Lemma 3.2. [Vu1, Lemma 2.41(2)], [AVV, Lemma 7.56] *Let $G \in \{\mathbb{B}^n, \mathbb{H}^n\}$, and let ρ_G stand for the respective hyperbolic metric. Then for all $x, y \in G$*

$$j_G(x, y) \leq \rho_G(x, y) \leq 2j_G(x, y).$$

The following theorem solves a question posed in [HKLV, Open problem 3.2].

Theorem 3.3. *Let G be a proper subdomain of \mathbb{R}^n . Then for all $x, y \in G$ we have*

$$s_G(x, y) \leq \frac{1}{\log 3} j_G(x, y)$$

and the constant $\frac{1}{\log 3} \approx 0.91$ is the best possible.

Proof. Let us fix the points x and y . By rescaling the domain we may assume that $|x - y| = 1$. We can also assume that $d(x) \leq d(y)$, because otherwise we can swap the points.

We denote $t = d(x) > 0$. Now

$$j_G(x, y) = \log \left(1 + \frac{1}{t} \right)$$

and we divide the proof into two cases: $t \leq \frac{1}{2}$ and $t > \frac{1}{2}$.

We assume first that $t \leq \frac{1}{2}$. Now $j_G(x, y) \geq \log 3$ and since $s_G(x, y) \leq 1$ we have

$$s_G(x, y) \leq 1 \leq \frac{j_G(x, y)}{\log 3}.$$

We assume then that $t > \frac{1}{2}$. We want to maximize $s_G(x, y)$ in terms of t . In other words, we want to find the smallest ellipsoid with foci x and y , which has at least one point outside the set $B^n(x, t) \cap B^n(y, t)$. Since $t > \frac{1}{2}$ the set $B^n(x, t) \cap B^n(y, t)$ is simply connected and the point z on the smallest ellipsoid can be found at $S^{n-1}(x, t) \cap S^{n-1}(y, t)$. Now

$$s_G(x, y) = \frac{|x - y|}{|x - z| + |y - z|} = \frac{1}{2t}$$

and we want to find a lower bound for the function

$$f(t) = \frac{j_G(x, y)}{s_G(x, t)} = 2t \log \left(1 + \frac{1}{t} \right), \quad t > \frac{1}{2}.$$

We can show that $g(t) = \frac{\log(1+t)}{t}$ is decreasing for t , because

$$\begin{aligned} g'(t) &= \frac{\frac{t}{1+t} - \log(1+t)}{t^2} \\ &\leq \frac{\frac{t}{1+t} - \frac{2t}{2+t}}{t^2} \leq 0 \end{aligned}$$

so it is increasing for $\frac{1}{t}$, thus $f(t)$ is increasing. We collect $f(t) > f(\frac{1}{2}) = \log 3$ and the claimed inequality is proved.

The constant $\frac{1}{\log 3}$ can be easily verified to be the best possible by investigating the domain $G = \mathbb{R}^n \setminus \{0\}$. For any $x \in G$ selecting $y = -x$ gives $s_G(x, y) = 1$ and $j_G(x, y) = \log 3$. \square

Lemma 3.4. *If $x, y \in G \subset \mathbb{R}^n$ and G is convex, then*

$$(3.5) \quad s_G(x, y) \leq p_G(x, y).$$

Here equality holds for all $x, y \in G$ if $G = \mathbb{H}^n$.

Proof. Suppose z that $z \in \partial G$ is an extremal boundary point for s -metric for which the equality holds in (1.1). We draw a line L through z tangent to ∂G . By geometry

$$|x - z| + |z - y| = |x - \bar{y}| = \sqrt{|x - y|^2 + 4d_1(x)d_1(y)},$$

$d_1(x) = d(x, L)$, $d_1(y) = d(y, L)$. Because G is convex it is clear that L is outside G , but $d(x), d(y)$ are the shortest distances from x, y to ∂G , so obviously $d(x) \leq d_1(x)$, $d(y) \leq d_1(y)$, thus

$$\begin{aligned} s_G(x, y) &= \frac{|x - y|}{|x - z| + |z - y|} \\ &= \frac{|x - y|}{\sqrt{|x - y|^2 + 4d_1(x)d_1(y)}} \\ &\leq \frac{|x - y|}{\sqrt{|x - y|^2 + 4d(x)d(y)}} \\ (3.6) \quad &= p_G(x, y). \end{aligned}$$

\square

Lemma 3.7. *For $x, y \in \mathbb{B}^n$ we have*

$$s_{\mathbb{B}^n}(x, y) \geq s_{\mathbb{B}^n}(x_s, y_s) = \frac{|x - y|}{\sqrt{|x - y|^2 + 4(1 - |m|)^2}},$$

where $m = \frac{x_1 + y_1}{2}$ and $x_1, y_1 \in \partial \mathbb{B}^n$ are the points of intersection of the line through x and y with $\partial \mathbb{B}^n$, $|x - y| = |x_s - y_s|$, and $|x_s| = |y_s|$ moreover

$$|m| = \frac{\sqrt{|x|^2|y|^2 - (x \cdot y)^2}}{|x - y|},$$

$$x_s = x_1 + \frac{y_1 - x_1}{|y_1 - x_1|} \left(\sqrt{1 - |m|^2} - \frac{|x - y|}{2} \right),$$

$$y_s = y_1 + \frac{x_1 - y_1}{|x_1 - y_1|} \left(\sqrt{1 - |m|^2} - \frac{|x - y|}{2} \right),$$

and hence

$$s_{\mathbb{B}^n}(x, y) \geq \frac{|x - y|^2}{|x - y|^4 + 4(|x - y| - \sqrt{|x|^2|y|^2 - (x \cdot y)^2})^2}.$$

Proof. If we move $x, y \in \mathbb{B}^n$ to $x_s, y_s \in \mathbb{B}^n$ which are symmetric with respect to midpoint m of the segment $[x_1, y_1]$, then we see easily that the extremal ellipse with foci x_s, y_s is larger than the extremal ellipse with foci x, y and hence by (1.1),

$$s_{\mathbb{B}^n}(x, y) \geq s_{\mathbb{B}^n}(x_s, y_s) = \frac{|x - y|}{\sqrt{|x - y|^2 + 4(1 - |m|)^2}}.$$

$|m|$ is the shortest distance from origin to the line \overline{xy} , which by the Law of Cosines, $|m| = \frac{\sqrt{|x|^2|y|^2 - (x \cdot y)^2}}{|x - y|}$, and therefore

$$s_{\mathbb{B}^n}(x_s, y_s) = \frac{|x - y|^2}{|x - y|^4 + 4(|x - y| - \sqrt{|x|^2|y|^2 - (x \cdot y)^2})^2},$$

and the proof is complete. \square

Lemma 3.8. For $x, y \in \mathbb{B}^n$ with $|x| > |y|$, $y_r = x - \frac{x}{|x|}|x - y| = -\frac{x}{|x|}(|x| - |x - y|)$,

$$s_{\mathbb{B}^n}(x, y) \geq s_{\mathbb{B}^n}(x, y_r) = \frac{|x - y|}{|x - y| + 2(1 - t)} \equiv w(x, y), \quad t = \max\{|x|, |y|\}.$$

Proof. Note that $y_r \in [x, -x]$ and $|x - y| = |x - y_r|$. By geometric properties of the ellipse it is clear that $s_{\mathbb{B}^n}(x, y) \geq s_{\mathbb{B}^n}(x, y_r)$ and thus

$$\begin{aligned} s_{\mathbb{B}^n}(x, y) &= \sup_{z \in \partial G} \frac{|x - y|}{|x - z| + |z - y|} \\ &\geq s_{\mathbb{B}^n}(x, y_r) \\ &= \frac{|x - y|}{|x - y| + 2(1 - t)}, \quad t = \max\{|x|, |y|\}. \end{aligned}$$

\square

Lemma 3.9. For all $x, y \in \mathbb{B}^n$ we have

$$(3.10) \quad s_{\mathbb{B}^n}(x, y) \leq p_{\mathbb{B}^n}(x, y) \leq \tanh \frac{\rho_{\mathbb{B}^n}(x, y)}{2} \leq 2p_{\mathbb{B}^n}(x, y).$$

Proof. The upper bound follows from Lemma 3.4 and Theorem 3.23. For the lower bound clearly

$$\begin{aligned} (1 - |x|^2)(1 - |y|^2) &= (1 - |x|)(1 - |y|)(1 + |x|)(1 + |y|) \\ &\leq 4(1 - |x|)(1 - |y|), \end{aligned}$$

so

$$\begin{aligned}
 \tanh \frac{\rho_{\mathbb{B}^n}(x, y)}{2} &= \frac{|x - y|}{\sqrt{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}} \\
 &\geq \frac{|x - y|}{\sqrt{|x - y|^2 + 4d(x)d(y)}} \\
 (3.11) \qquad &= p_{\mathbb{B}^n}(x, y).
 \end{aligned}$$

□

Theorem 3.12. *If $z \in G$, $r = d(z)$, $0 < \lambda < 1$, $x, y \in \mathbb{B}^n(z, \lambda r)$, then*

$$s_G(x, y) \leq \left(\frac{1 + \lambda}{1 - \lambda} \right) p_G(x, y).$$

Proof. By monotonicity of s -metric

$$s_G(x, y) \leq s_{\mathbb{B}^n(z, r)}(x, y) \leq p_{\mathbb{B}^n(z, r)}(x, y) \leq \frac{|x - y|}{\sqrt{|x - y|^2 + 4(1 - \lambda)^2 r^2}}.$$

If $x, y \in \mathbb{B}^n(z, \lambda r)$, we easily see that

$$(3.13) \qquad (1 - \lambda)r < d_G(x) < (1 + \lambda)r.$$

Now if we choose $c = \left(\frac{1 + \lambda}{1 - \lambda} \right)$, then

$$\frac{|x - y|}{\sqrt{|x - y|^2 + 4(1 - \lambda)^2 r^2}} \leq \frac{c|x - y|}{\sqrt{|x - y|^2 + 4(1 + \lambda)^2 r^2}} \leq cp_G(x, y).$$

□

Theorem 3.14. *If $z \in G$, $0 < \lambda < 1$, $x, y \in \mathbb{B}^n(z, \lambda d(z))$, then*

$$s_{\mathbb{B}^n(z, d(z))}(x, y) \leq C j_{\mathbb{B}^n(z, d(z))}(x, y), \quad C = \frac{2(1 - \lambda)}{1 + 2\lambda}.$$

Proof. From $x, y \in \mathbb{B}^n(z, \lambda d(z))$ it follows that

$$(3.15) \qquad \frac{|x - y|}{d(z)} \leq 2\lambda.$$

Because for all $x, y \in \mathbb{B}^n(z, \lambda d(z))$,

$$|x - w| + |y - w| \geq 2(1 - \lambda)d(z),$$

we see that

$$s_{\mathbb{B}^n(z, d(z))}(x, y) \leq \frac{|x - y|}{2(1 - \lambda)d(z)}.$$

and by $\log(1 + t) > \frac{2t}{2+t}$, and (3.15) we see that

$$\begin{aligned}
 j_{\mathbb{B}^n(z, d(z))}(x, y) &\geq \log \left(1 + \frac{|x - y|}{(1 + \lambda)d(z)} \right) \\
 &\geq \frac{\frac{2|x - y|}{(1 + \lambda)d(z)}}{2 + \frac{|x - y|}{(1 + \lambda)d(z)}} \\
 &\geq \frac{|x - y|}{(1 + 2\lambda)d(z)},
 \end{aligned}$$

So it suffices to choose $C = \frac{2(1 - \lambda)}{1 + 2\lambda}$.

□

Corollary 3.16. *Under the assumptions of Theorem 3.14,*

$$j_{\mathbb{B}^n(z, d(z))}(x, y) \leq \frac{2(1+\lambda)}{1-\lambda} s_{\mathbb{B}^n(z, d(z))}(x, y).$$

Proof. By (3.13),

$$\begin{aligned} j_{\mathbb{B}^n(z, d(z))}(x, y) &\leq \log \left(1 + \frac{|x-y|}{(1-\lambda)(d(z))} \right) \\ &\leq \frac{|x-y|}{(1-\lambda)(d(z))}. \end{aligned}$$

On the other hand

$$\begin{aligned} s_{\mathbb{B}^n(z, d(z))}(x, y) &\geq \frac{|x-y|}{2 \min\{1-|x|, 1-|y|\} + |x-y|} \\ &\geq \frac{|x-y|}{(2\lambda+2)d(z)} \\ &\geq \frac{|x-y|}{2(1+\lambda)d(z)}, \end{aligned}$$

now it suffices to find C such that

$$\frac{|x-y|}{2(1+\lambda)d(z)} \geq C \frac{|x-y|}{(1-\lambda)(d(z))},$$

so $C = \frac{2(1+\lambda)}{1-\lambda}$, and the proof is complete. □

Theorem 3.17. *If $z \in G$, $0 < \lambda < 1$, $x, y \in \mathbb{B}^n(z, \lambda d(z))$, then*

$$j_G(x, y) \leq C p_G(x, y), \quad C = \frac{2}{1-\lambda}.$$

Proof. By symmetry we may assume that $d(x) \leq d(y)$. Then by $\log(1+t) \leq t$, $t > 0$ we have

$$j_G(x, y) \leq \frac{|x-y|}{\min\{d(x), d(y)\}} = \frac{|x-y|}{d(x)}.$$

On the other hand by the assumption we get $d(z) \leq \frac{1}{1-\lambda} \min\{d(x), d(y)\}$, and

$$\begin{aligned} \frac{1-\lambda}{1+\lambda} &\leq \frac{d(x)}{d(y)} \leq \frac{1+\lambda}{1-\lambda}, \\ p_G(x, y) &= \frac{|x-y|}{\sqrt{|x-y|^2 + 4d(x)d(y)}} \\ &\geq \frac{|x-y|}{\sqrt{\left(2\lambda \frac{d(x)}{1-\lambda}\right)^2 + 4d(x) \frac{1+\lambda}{1-\lambda} d(x)}} \\ &\geq \frac{1-\lambda}{2} \cdot \frac{|x-y|}{d(x)}. \end{aligned}$$

We see that

$$j_G(x, y) \leq \frac{|x-y|}{d(x)} \leq C \frac{1-\lambda}{2} \cdot \frac{|x-y|}{2} \leq C p_G(x, y),$$

holds if $C \geq \frac{2}{1-\lambda}$, and the proof is complete. \square

Theorem 3.18. *If $x, y \in G$, then*

$$p_G(x, y) \leq C j_G(x, y),$$

$$C = \max \left\{ \frac{2+\mu}{4}, \frac{1}{\log(1+\mu)} \right\}, \quad 0 < \mu < 1.$$

Proof. Suppose that $j_G(x, y) \leq \log(1+\mu)$. By symmetry we may assume that $d(x) \leq d(y)$. Then $p_G(x, y) \leq \frac{|x-y|}{2d(x)}$. On the other hand, by the assumption

$$\begin{aligned} j_G(x, y) &\geq \log \left(1 + \frac{|x-y|}{d(x)} \right) \\ &\geq \frac{2 \frac{|x-y|}{d(x)}}{2+\mu}, \\ &= \frac{2}{2+\mu} \cdot \frac{|x-y|}{d(x)}. \end{aligned}$$

So the inequality

$$p_G(x, y) \leq \frac{|x-y|}{2d(x)} \leq C \frac{2}{2+\mu} \cdot \frac{|x-y|}{d(x)},$$

holds if $C \geq \frac{2+\mu}{4}$.

In the remaining case $j_G(x, y) \geq \log(1+\mu)$, and hence

$$p_G(x, y) \leq 1 \leq C \log(1+\mu) \leq C j_G(x, y),$$

when $C \geq \frac{1}{\log(1+\mu)}$.

In both cases we may choose $C = \max \left\{ \frac{2+\mu}{4}, \frac{1}{\log(1+\mu)} \right\}$. \square

Proof of Theorem 1.10 The result follows from Theorems 3.3 and 3.18. \square

Theorem 3.19. (1) *For $x, y \in \mathbb{B}^2$ we have*

$$v_{\mathbb{B}^2}(x, y) \leq 2j_{\mathbb{B}^2}(x, y).$$

(2) *If $\lambda \in (0, 1)$ and $x, y \in \mathbb{B}^2(\lambda)$ then*

$$\frac{3(1-\lambda^2)}{2(3+\lambda^2)} j_{\mathbb{B}^2}(x, y) \leq v_{\mathbb{B}^2}(x, y).$$

Proof. (1) By [KLVW, 3.12] we have $v_{\mathbb{B}^2}(x, y) \leq \rho_{\mathbb{B}^2}(x, y)$. Now the proof follows by Lemma 3.2.

(2)

$$\sinh \frac{\rho_{\mathbb{B}^2}(x, y)}{2} \leq \sinh j_{\mathbb{B}^2}(x, y) \leq \sinh \left(\log \left(1 + \frac{2\lambda}{1-\lambda} \right) \right) = \frac{2\lambda}{1-\lambda^2},$$

by [KLVW, 3.15] $\rho_{\mathbb{B}^2}^* \leq v_{\mathbb{B}^2} \leq 2\rho_{\mathbb{B}^2}^*$, where

$$\rho_{\mathbb{B}^2}^*(x, y) = \arctan \left(\sinh \frac{\rho_{\mathbb{B}^2}(x, y)}{2} \right),$$

so by [DC]

$$\begin{aligned}
\frac{3x}{1+2\sqrt{1+x^2}} &< \arctan x < \frac{2x}{1+\sqrt{1+x^2}}, \\
\rho_{\mathbb{B}^2}^*(x, y) &= \arctan \left(\sinh \frac{\rho_{\mathbb{B}^2}(x, y)}{2} \right) \\
&\geq \frac{3 \sinh \frac{\rho_{\mathbb{B}^2}(x, y)}{2}}{1+2\sqrt{1+\sinh^2 \frac{\rho_{\mathbb{B}^2}(x, y)}{2}}} \\
&\geq \frac{3 \sinh \frac{j_{\mathbb{B}^2}(x, y)}{2}}{1+2\sqrt{1+\left(\frac{2\lambda}{1-\lambda^2}\right)^2}} \\
&= \frac{3(1-\lambda^2)}{3+\lambda^2} \sinh \frac{j_{\mathbb{B}^2}(x, y)}{2} \\
&\geq \frac{3(1-\lambda^2)}{2(3+\lambda^2)} j_{\mathbb{B}^2}(x, y).
\end{aligned}$$

Thus

$$\frac{3(1-\lambda^2)}{2(3+\lambda^2)} j_{\mathbb{B}^2}(x, y) \leq v_{\mathbb{B}^2}(x, y).$$

□

Theorem 3.20. *If $x, y, z \in G$, $\lambda \in (0, 1)$ then $p_G(x, y) \leq \frac{1+\lambda}{1-\lambda} s_G(x, y)$, for $x, y \in \mathbb{B}^n(z, \lambda d(z))$.*

Proof. Fix $w \in \partial G \cap S^{n-1}(z, d(z))$. Thus

$$s_G(x, y) \geq \frac{|x-y|}{|x-w|+|y-w|} \geq \frac{|x-y|}{2(1+\lambda)d(z)},$$

On the other hand because $d(x) \geq (1-\lambda)d(z)$

$$p_G(x, y) = \frac{|x-y|}{\sqrt{|x-y|^2 + 4d(x)d(y)}} \leq \frac{|x-y|}{2\sqrt{(1-\lambda)^2 d(z)^2}} = \frac{|x-y|}{2(1-\lambda)d(z)},$$

we see that

$$p_G(x, y) \leq \frac{|x-y|}{2(1-\lambda)d(z)} \leq C \frac{|x-y|}{2(1+\lambda)d(z)} \leq C s_G(x, y),$$

holds if $C \geq \frac{1+\lambda}{1-\lambda}$.

□

Theorem 3.21. *Let $0 < \lambda < 1$, $x, y \in \mathbb{B}^2(\lambda)$. Then*

(1)

$$s_{\mathbb{B}^2}(x, y) \leq \frac{4(3+\lambda^2)}{3(1+2\lambda)(1+\lambda)} v_{\mathbb{B}^2}(x, y),$$

(2)

$$v_{\mathbb{B}^2}(x, y) \leq \frac{4(1+\lambda)}{1-\lambda} s_{\mathbb{B}^2}(x, y).$$

Proof. (1) By Theorems 3.14 and 3.19,

$$s_{\mathbb{B}^2}(x, y) \leq \frac{4(3 + \lambda^2)}{3(1 + 2\lambda)(1 + \lambda)} v_{\mathbb{B}^2}(x, y).$$

(2) By Theorems 3.19 and 3.16,

$$v_{\mathbb{B}^2}(x, y) \leq 2j_{\mathbb{B}^2}(x, y) \leq \frac{4(1 + \lambda)}{1 - \lambda} s_{\mathbb{B}^2}(x, y).$$

□

Theorem 3.22. (1) If $\lambda \in (0, 1)$ and $x, y \in \mathbb{B}^2(\lambda)$ then

$$v_{\mathbb{B}^2}(x, y) \leq \frac{4}{(1 - \lambda)} p_{\mathbb{B}^2}(x, y).$$

(2) If $x, y \in \mathbb{B}^2$ with $v_{\mathbb{B}^2}(x, y) \in (0, \pi/2)$, then

$$p_{\mathbb{B}^2}(x, y) \leq 2v_{\mathbb{B}^2}(x, y),$$

Proof. (1) By Theorems 3.21 and 3.4,

$$v_{\mathbb{B}^2}(x, y) \leq \frac{4}{(1 - \lambda)} s_{\mathbb{B}^2}(x, y) \leq \frac{4}{(1 - \lambda)} p_{\mathbb{B}^2}(x, y).$$

(2) By Lemma 3.9 and [KLVW, 3.15] we have

$$\rho_{\mathbb{B}^2}^*(x, y) = \arctan \left(\sinh \frac{\rho_{\mathbb{B}^2}(x, y)}{2} \right) \leq v_{\mathbb{B}^2}(x, y).$$

Then

$$\rho_{\mathbb{B}^2}(x, y) \leq 2 \operatorname{arsinh}(\tan(v_{\mathbb{B}^2}(x, y))).$$

Then if $v_{\mathbb{B}^2}(x, y) \in (0, \pi/2)$,

$$\begin{aligned} p_{\mathbb{B}^2}(x, y) &\leq \tanh(\operatorname{arsinh}(\tan(v_{\mathbb{B}^2}(x, y)))) \\ &= \frac{\tan(v_{\mathbb{B}^2}(x, y))}{\sqrt{1 + \tan^2(v_{\mathbb{B}^2}(x, y))}} \\ &= \sin(\tan(v_{\mathbb{B}^2}(x, y))) \\ &\leq \tan(v_{\mathbb{B}^2}(x, y)) \\ &\leq 2v_{\mathbb{B}^2}(x, y). \end{aligned}$$

□

Theorem 3.23. For $x, y \in \mathbb{B}^n$ we have

$$(3.24) \quad \tanh \left(\frac{\rho_{\mathbb{B}^n}(x, y)}{2} \right) \leq 2s_{\mathbb{B}^n}(x, y).$$

Proof. By (2.3) and (1.1) it is enough to show that

$$I \leq 2|x||x^* - y|, \quad I = \inf_{z \in \partial \mathbb{B}^n} |x - z| + |z - y|,$$

Assume $|y| \leq |x|$. Denote $|y| = t|x|$ for $t \in [0, 1]$, $\gamma \in [0, \pi]$, is angle between $[0, x]$ and $[0, y]$.

Case A. $\gamma \geq \frac{\pi}{2}$. Now

$$(3.25) \quad 2|x||x^* - y| \geq 2|x| \frac{1}{|x|} = 2,$$

Moreover choose $z_1 = \frac{x}{|x|}$, then

$$\begin{aligned}
 (3.26) \quad I &\leq |x - z_1| + |z_1 - y| \\
 &\leq 1 - |x| + \sqrt{t^2|x|^2 + 1 + 2t|x|} \\
 &= 2 - |x| + t|x| \\
 &= 2 - (|x|(1 - t)) \leq 2.
 \end{aligned}$$

So by (3.25) and (3.26),

$$I \leq 2|x||x^* - y|,$$

Case B. $\gamma \leq \frac{\pi}{2}$.

$$(3.27) \quad 2|x||x^* - y| = 2|y|x - z_2| = 2|x|y - z_1|,$$

where $|z_2| = \frac{y}{|y|}$ and $|z_1| = \frac{x}{|x|}$. Next we choose z in the infimum to be the middle point of z_1 and z_2 on the unit sphere. This means that $\angle(x, 0, z) = \angle(z, 0, y) = \gamma/2$ and $|z| = 1$. We know that

$$I \leq |x - z| + |z - y|,$$

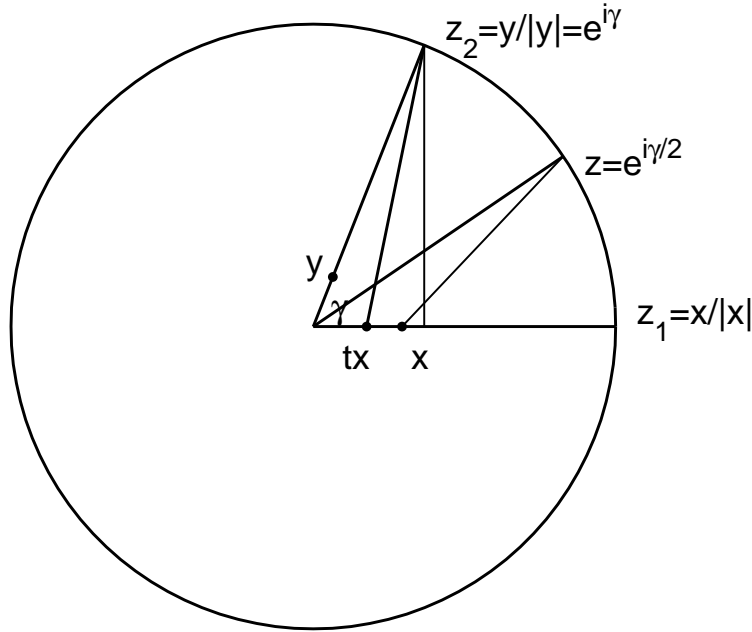


FIGURE 2. Proof of Theorem 3.23. The case $r = |z - x| > \sin(\gamma)$.

We next show that

$$(3.28) \quad p/r \geq 1, \quad p = |z_2 - |y|x|, \quad r = |z - x|.$$

By elementary geometry, applying the properties of the right triangle $\Delta(0, z_2, (\cos \gamma)z_1)$ and the Law of Cosines, we see that

$$\begin{aligned}
 (3.29) \quad p &\geq |z_2 - (\cos \gamma)z_1| = \sin \gamma \geq \sqrt{1 + \cos^2(\gamma) - 2\cos(\gamma)\cos(\gamma/2)} = |z - (\cos \gamma)z_1|.
 \end{aligned}$$

The second inequality follows because for $\gamma \in (0, \pi/2)$,

$$\sin^2(\gamma) > 1 + \cos^2(\gamma) - 2\cos(\gamma)\cos(\gamma/2)$$

by basic trigonometry.

If $r \leq \sin \gamma$, then by (3.29) $p/r \geq 1$ clearly holds. In the remaining case $r = |z - x| > \sin \gamma$. Because $x \in [0, z_1]$ and this means by (3.29) that $x \in [0, (\cos \gamma)z_1]$ and hence the angle between the segments $[x, z_2]$ and $[x, 0]$ is more than $\pi/2$ and hence

$$p = |z_2 - |y|x| > |z_2 - x|.$$

Finally, we see that $p/r \geq |z_2 - x|/|z - x| > 1$, because x and z both are in the same half plane determined by the bisecting normal of the segment $[z_2, z]$. Symmetrically we obtain that

$$|z - y| \leq ||x|y - z_1|,$$

and hence

$$|x - z| + |z - y| \leq ||y|x - z_2| + ||x|y - z_1| = 2|x||x^* - y|$$

and the proof is complete. \square

Corollary 3.30. (1) If $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is a Möbius transformation onto \mathbb{H}^n , then for all $x, y \in \mathbb{H}^n$,

$$(3.31) \quad s_{\mathbb{H}^n}(f(x), f(y)) = s_{\mathbb{H}^n}(x, y).$$

(2) If $f : \mathbb{H}^n \rightarrow \mathbb{B}^n$ is a Möbius transformation onto \mathbb{B}^n , then for all $x, y \in \mathbb{H}^n$,

$$(3.32) \quad s_{\mathbb{B}^n}(f(x), f(y)) \leq s_{\mathbb{H}^n}(x, y).$$

(3) If $f : \mathbb{B}^n \rightarrow \mathbb{H}^n$ is a Möbius transformation onto \mathbb{H}^n , then for all $x, y \in \mathbb{B}^n$,

$$(3.33) \quad s_{\mathbb{H}^n}(f(x), f(y)) \leq 2s_{\mathbb{B}^n}(x, y).$$

(4) If $f : \mathbb{B}^n \rightarrow \mathbb{B}^n$ is a Möbius transformation onto \mathbb{B}^n , then for all $x, y \in \mathbb{B}^n$,

$$(3.34) \quad s_{\mathbb{B}^n}(f(x), f(y)) \leq 2s_{\mathbb{B}^n}(x, y).$$

Proof. It is a basic fact that a Möbius transformation $f : G \rightarrow D = fG$ with $G, D \in \{\mathbb{B}^n, \mathbb{H}^n\}$ defines an isometry $f : (G, \rho_G) \rightarrow (D, \rho_D)$ between hyperbolic spaces. This fact combined with (2.5), Lemma 3.9 and Theorem 3.23 yields the proof. \square

We were led to Conjecture 1.9 by MATLAB experiments. We now show that if the conjecture holds true, then the constant $1+|a|$ cannot be improved when $n = 2$.

Proof of Theorem 1.8. Observe first that for $0 < a < b < 1$ we have

$$s_{\mathbb{B}^n}(ae_1, be_1) = \frac{b-a}{2-a-b}.$$

Let $T_a : \mathbb{B}^n \rightarrow \mathbb{B}^n$ be the Möbius map with $T_a(ae_1) = 0$. Then choose $a \in (0, 1)$, $b \in (a, 1)$, $x = ae_1$ and $y = be_1$. Now we have $s_{\mathbb{B}^n}(x, y) = \frac{b-a}{2-a-b}$, and $T_a(b) = \frac{b-a}{1-ab}e_1 = ce_1$.

$$\begin{aligned} s_{\mathbb{B}^n}(T_a(a), T_a(b)) &= s(0, c) = \frac{c}{2-c} \\ &= \frac{b-a}{2+a-b-2ab}. \end{aligned}$$

Now

$$R = \frac{s_{\mathbb{B}^n}(a, b)}{s_{\mathbb{B}^n}(T_a(a), T_a(b))} = \frac{2+a-b-2ab}{2-a-b}.$$

Denote $b = \frac{1+ca}{1+c}$ for $c \geq 1$. Now

$$R = 1 + \frac{2ac}{1+2c},$$

if $a \rightarrow 1$ and $c \rightarrow \infty$ then $R \rightarrow 2$, and if $c \rightarrow \infty$ then $R \rightarrow 1+a$. Hence

$$\sup \left(\frac{s_{\mathbb{B}^n}(x, y)}{s_{\mathbb{B}^n}(T_a(x), T_a(y))} : x, y \in \mathbb{B}^n \right) \geq 1 + |a|. \quad \square$$

Theorem 3.35. *If $f : \mathbb{B}^n \rightarrow \mathbb{B}^n = f(\mathbb{B}^n)$ is a Möbius transformation with $f(a) = 0$, then for all distinct points $x, y \in \mathbb{B}^n$, then we have*

$$s_{\mathbb{B}^n}(f(x), f(y)) \leq \frac{1+|a|}{1-|a|} s_{\mathbb{B}^n}(x, y).$$

Proof. If $f(0) = 0$ then f is a rotation and there is nothing to prove. Otherwise $f(a) = 0$ some $a \neq 0$. Let $f = T_a$ be the canonical representation of a Möbius transformation, see [B]. Then with $a^* = a/|a|^2$, $r = \sqrt{|a|^{-2} - 1}$ we have

$$|T_a(x) - T_a(y)| = \frac{r^2|x-y|}{|x-a^*||y-a^*|}.$$

If $w \in \partial\mathbb{B}^n$, then this formula yields

$$Q(x, y, w) = \frac{|T_ax - T_ay|}{|T_ax - T_aw| + |T_aw - T_ay|} : \frac{|x-y|}{|x-w| + |w-y|} = \frac{|x-w| + |w-y|}{\beta|x-w| + \gamma|w-y|}$$

with $\beta = |y-a^*|/|w-a^*|$, $\gamma = |x-a^*|/|w-a^*|$. Clearly,

$$|w-a^*| \leq 1+|a|^{-1} \quad |x-a^*|, |y-a^*| \geq |a|^{-1} - 1$$

and hence

$$Q(x, y, w) \leq \frac{|x-w| + |w-y|}{|x-w| + |w-y|} \frac{1+|a|}{1-|a|} = \frac{1+|a|}{1-|a|}.$$

Thus we have for all $x, y \in \mathbb{B}^n$, $w \in \partial\mathbb{B}^n$

$$\frac{|T_ax - T_ay|}{|T_ax - T_aw| + |T_aw - T_ay|} \leq \frac{1+|a|}{1-|a|} \frac{|x-y|}{|x-w| + |w-y|}.$$

Taking supremum over all $w \in \partial\mathbb{B}^n$ yields the desired conclusion. \square

We compare next j , p , s and v in domains $\mathbb{R}^n \setminus \{e_1\}$ and \mathbb{B}^n . By the monotonicity with respect to domains it is clear that for all $x, y \in \mathbb{B}^n$ and $m \in \{j, p, s, v\}$ we have $m_{\mathbb{R}^n \setminus \{e_1\}}(x, y) \leq m_{\mathbb{B}^n}(x, y)$. Next we consider the comparison in the opposite direction. Let us start by introducing the following lemma.

Lemma 3.36. *For $0 < b \leq 1 \leq a$ the function*

$$f(x) = \frac{\log(1+ax)}{\log(1+bx)}, x \in (0, \infty),$$

is decreasing.

Proof. Since

$$f'(x) = \frac{\frac{a}{1+ax} \log(1+bx) - \frac{b}{1+bx} \log(1+ax)}{\log^2(1+bx)}$$

the inequality $f'(x) \leq 0$ is equivalent to

$$(3.37) \quad \frac{1+bx}{b} \log(1+bx) \leq \frac{1+ax}{a} \log(1+ax).$$

Now we show that the function

$$g(c) = \frac{1+cx}{c} \log(1+cx)$$

is increasing on $(0, \infty)$, which implies (3.37) and the assertion. This is clear because

$$g'(c) = \frac{cx - \log(1+cx)}{c^2}$$

and $g'(c) > 0$ as $\log(1+y) < y$ for $y > 0$. \square

Theorem 3.38. *Let $t \in (0, 1)$ and $m \in \{j, p, s\}$. There exists a constant $c_m = c_m(t) > 1$ such that for all $x, y \in \mathbb{B}^n$ with $|x|, |y| < t$ we have*

$$m_{\mathbb{B}^n}(x, y) \leq c_m m_{\mathbb{R}^n \setminus \{e_1\}}(x, y).$$

Moreover, $c(t) \rightarrow 1$ as $t \rightarrow 0$ and $c(t) \rightarrow \infty$ as $t \rightarrow 1$.

Proof. We denote $m_1 = m_{\mathbb{B}^n}$, $m_2 = m_{\mathbb{R}^n \setminus \{e_1\}}$ and find upper bound for $\frac{m_1}{m_2}$, which gives us c_m .

Let us start with $m = j$. We denote $z = |x - y| \in [0, 2t)$ and obtain by Lemma 3.36

$$\begin{aligned} \frac{j_1}{j_2} &= \frac{\log\left(1 + \frac{z}{\min\{1-|x|, 1-|y|\}}\right)}{\log\left(1 + \frac{z}{\min\{|x-e_1|, |y-e_1|\}}\right)} \leq \frac{\log\left(1 + \frac{z}{1-t}\right)}{\log\left(1 + \frac{z}{1+t}\right)} \\ &\leq \lim_{z \rightarrow 0} \frac{\log\left(1 + \frac{z}{1-t}\right)}{\log\left(1 + \frac{z}{1+t}\right)} = \lim_{z \rightarrow 0} \frac{1+t+z}{1-t+z} = \frac{1+t}{1-t} = c_j, \end{aligned}$$

where the second equality follows from l'Hôpital's rule. Obviously $c_j \rightarrow 1$ as $t \rightarrow 0$ and $c_j \rightarrow \infty$ as $t \rightarrow 1$.

Let us now consider $m = p$. Now

$$\frac{p_1^2}{p_2^2} = \frac{|x-y|^2 + 4|x-e_1||y-e_1|}{|x-y|^2 + 4(1-|x|)(1-|y|)} \leq \frac{4t^2 + 4(1+t)^2}{0 + 4(1-t)^2} = \frac{2t^2 + 2t + 1}{t^2 - 2t + 1}$$

and we can choose

$$c_p = \sqrt{\frac{2t^2 + 2t + 1}{t^2 - 2t + 1}}.$$

Clearly $c_p \rightarrow 1$ as $t \rightarrow 0$ and $c_p \rightarrow \infty$ as $t \rightarrow 1$.

Next we set $m = s$ and obtain by geometry

$$\frac{s_1}{s_2} = \frac{|x - e_1| + |y - e_1|}{\inf_{z \in \partial \mathbb{B}^n} |x - z| + |z - y|} \leq \frac{2(1+t)}{2(1-t)} = \frac{1+t}{1-t} = c_s.$$

Again it is clear that $c_s \rightarrow 1$ as $t \rightarrow 0$ and $c_s \rightarrow \infty$ as $t \rightarrow 1$. \square

Note that for the visual angle metric v the result of Theorem 3.38 does not hold. We would need an upper bound for

$$\frac{v_{\mathbb{B}^n}(x, y)}{v_{\mathbb{R}^n \setminus \{e_1\}}(x, y)} = \frac{\sup_{z \in \partial \mathbb{B}^n} \angle(x, z, y)}{\angle(x, e_1, y)},$$

but choosing x and y to be distinct points on the x_1 -axis

$$\sup_{z \in \partial \mathbb{B}^n} \angle(x, z, y) > 0$$

and $\angle(x, e_1, y) = 0$.

Next result demonstrates the sensitivity to boundary variation. We consider domains $G \subset \mathbb{R}^n$ and $G' = G \setminus \{x\}$, where $x \in G$. Again by the monotonicity we have $m_G(y, z) \leq m_{G'}(y, z)$ for all $y, z \in G'$ and $m \in \{j, p, s, v\}$.

Theorem 3.39. *Let $G \subset \mathbb{R}^n$, $x \in G$, $t \in (0, 1)$ and $m \in \{j, p, s\}$. Then there exists a constant $c_m = c_m(t)$ such that for all $y, z \in G \setminus B(x, td_G(x))$ we have*

$$m_{G \setminus \{x\}}(y, z) \leq c_m m_G(y, z).$$

Moreover, the constant is best possible as $t \rightarrow 1$. This means that $c_j, c_p, c_s \rightarrow 2$ as $t \rightarrow 1$.

Proof. We denote $G' = G \setminus \{x\}$ and will find an upper bound for $\frac{m_{G'}(y, z)}{m_G(y, z)}$.

We consider first $m = j$. If $d_G(y) = d_{G'}(y)$ and $d_G(z) = d_{G'}(z)$, then there is nothing to prove as $j_{G'}(y, z) = j_G(y, z)$ and we can choose $c_j = 1$. We consider next two cases: $d_G(y) \neq d_{G'}(y)$, $d_G(z) = d_{G'}(z)$ and $d_G(y) \neq d_{G'}(y)$, $d_G(z) \neq d_{G'}(z)$.

Let us assume $d_G(y) \neq d_{G'}(y)$ and $d_G(z) = d_{G'}(z)$ (or by symmetry we could as well assume $d_G(y) = d_{G'}(y)$ and $d_G(z) \neq d_{G'}(z)$). Now

$$\frac{j_{G'}(y, z)}{j_G(y, z)} = \frac{\log \left(1 + \frac{|y-z|}{\min\{d_{G'}(y), d_{G'}(z)\}} \right)}{\log \left(1 + \frac{|y-z|}{\min\{d_G(y), d_G(z)\}} \right)} = \frac{\log \left(1 + \frac{|y-z|}{\min\{|y-x|, d_G(z)\}} \right)}{\log \left(1 + \frac{|y-z|}{\min\{d_G(y), d_G(z)\}} \right)}.$$

Let us assume that $d_G(z) \leq d_G(y)$. If $d_G(z) \leq |y-x|$ then $j_{G'}(y, z) = j_G(y, z)$ and there is nothing to prove. If $d_G(z) \geq |y-x|$ then

$$\begin{aligned} \frac{j_{G'}(y, z)}{j_G(y, z)} &= \frac{\log \left(1 + \frac{|y-z|}{|y-x|} \right)}{\log \left(1 + \frac{|y-z|}{d_G(z)} \right)} \leq \frac{\log \left(1 + \frac{|y-z|}{td_G(x)} \right)}{\log \left(1 + \frac{|y-z|}{d_G(z)} \right)} \\ &\leq \frac{\log \left(1 + \frac{|y-z|}{td_G(x)} \right)}{\log \left(1 + \frac{|y-z|}{d_G(y)} \right)} \leq \frac{\log \left(1 + \frac{|y-z|}{td_G(x)} \right)}{\log \left(1 + \frac{|y-z|}{|y-x| + d_G(x)} \right)}. \end{aligned}$$

If $|x - y| \leq d_G(x)$ we have by Lemma 3.36

$$\begin{aligned} \frac{j_{G'}(y, z)}{j_G(y, z)} &\leq \frac{\log\left(1 + \frac{|y-z|}{td_G(x)}\right)}{\log\left(1 + \frac{|y-z|}{2d_G(x)}\right)} \leq \lim_{|y-z|/d_G(x) \rightarrow 0} \frac{\log\left(1 + \frac{|y-z|}{td_G(x)}\right)}{\log\left(1 + \frac{|y-z|}{2d_G(x)}\right)} \\ &\leq \lim_{|y-z|/d_G(x) \rightarrow 0} \frac{2 + \frac{|y-z|}{d_G(x)}}{t + \frac{|y-z|}{d_G(x)}} = \frac{2}{t}. \end{aligned}$$

If $|x - y| \geq d_G(x)$ again by Lemma 3.36

$$\begin{aligned} \frac{j_{G'}(y, z)}{j_G(y, z)} &\leq \frac{\log\left(1 + \frac{|y-z|}{|y-x|}\right)}{\log\left(1 + \frac{|y-z|}{2|y-x|}\right)} \leq \lim_{|y-z|/|y-x| \rightarrow 0} \frac{\log\left(1 + \frac{|y-z|}{|y-x|}\right)}{\log\left(1 + \frac{|y-z|}{2|y-x|}\right)} \\ &\leq \lim_{|y-z|/|y-x| \rightarrow 0} \frac{2 + \frac{|y-z|}{|y-x|}}{1 + \frac{|y-z|}{|y-x|}} = 2. \end{aligned}$$

Let us then assume $d_G(y) \leq d_G(z)$. Now $d_G(y) \neq d_{G'}(y)$ implies $|y - x| < d_G(y)$ and thus

$$(3.40) \quad \frac{j_{G'}(y, z)}{j_G(y, z)} = \frac{\log\left(1 + \frac{|y-z|}{|y-x|}\right)}{\log\left(1 + \frac{|y-z|}{d_G(y)}\right)} \leq \frac{\log\left(1 + \frac{|y-z|}{|y-x|}\right)}{\log\left(1 + \frac{|y-z|}{|y-x| + d_G(x)}\right)}.$$

If $|x - y| \leq d_G(x)$ we have by (3.40) and Lemma 3.36

$$\begin{aligned} \frac{j_{G'}(y, z)}{j_G(y, z)} &\leq \frac{\log\left(1 + \frac{|y-z|}{td_G(x)}\right)}{\log\left(1 + \frac{|y-z|}{2d_G(x)}\right)} \leq \lim_{|y-z|/d_G(x) \rightarrow 0} \frac{\log\left(1 + \frac{|y-z|}{td_G(x)}\right)}{\log\left(1 + \frac{|y-z|}{2d_G(x)}\right)} \\ &\leq \lim_{|y-z|/d_G(x) \rightarrow 0} \frac{2 + \frac{|y-z|}{d_G(x)}}{t + \frac{|y-z|}{d_G(x)}} = \frac{2}{t}. \end{aligned}$$

If $d_G(x) \leq |x - y|$ we have by (3.40) and Lemma 3.36

$$\begin{aligned} \frac{j_{G'}(y, z)}{j_G(y, z)} &\leq \frac{\log\left(1 + \frac{|y-z|}{|y-x|}\right)}{\log\left(1 + \frac{|y-z|}{2|y-x|}\right)} \leq \lim_{|y-z|/|y-x| \rightarrow 0} \frac{\log\left(1 + \frac{|y-z|}{|y-x|}\right)}{\log\left(1 + \frac{|y-z|}{2|y-x|}\right)} \\ &\leq \lim_{|y-z|/|y-x| \rightarrow 0} \frac{2 + \frac{|y-z|}{|y-x|}}{1 + \frac{|y-z|}{|y-x|}} = 2. \end{aligned}$$

Let us then assume $d_G(y) \neq d_{G'}(y)$ and $d_G(z) \neq d_{G'}(z)$. Now we may assume by symmetry that $|y - x| \leq |z - x|$ and thus

$$\frac{j_{G'}(y, z)}{j_G(y, z)} = \frac{\log\left(1 + \frac{|y-z|}{|y-x|}\right)}{\log\left(1 + \frac{|y-z|}{\min\{d_G(y), d_G(z)\}}\right)} \leq \frac{\log\left(1 + \frac{|y-z|}{|y-x|}\right)}{\log\left(1 + \frac{|y-z|}{|y-x| + d_G(x)}\right)}$$

and this is exactly the same as (3.40) so we know that it is $\leq \frac{2}{t}$.

Putting all this together gives us $c_j = \frac{2}{t}$.

Let now $m = p$. If $d_G(y) = d_{G'}(y)$ and $d_G(z) = d_{G'}(z)$, then there is nothing to prove as $p_{G'}(y, z) = p_G(y, z)$ and we can choose $c_p = 1$. We consider next two cases: $d_G(y) \neq d_{G'}(y)$, $d_G(z) = d_{G'}(z)$ and $d_G(y) \neq d_{G'}(y)$, $d_G(z) \neq d_{G'}(z)$.

Let us assume $d_G(y) \neq d_{G'}(y)$ and $d_G(z) = d_{G'}(z)$ (or by symmetry we could as well assume $d_G(y) = d_{G'}(y)$ and $d_G(z) \neq d_{G'}(z)$). Now

$$\begin{aligned} \frac{p_{G'}^2(y, z)}{p_G^2(y, z)} &= \frac{|y - z|^2 + 4d_G(y)d_G(z)}{|y - z|^2 + 4d_{G'}(y)d_{G'}(z)} = \frac{|y - z|^2 + 4d_G(y)d_G(z)}{|y - z|^2 + 4|y - x|d_G(z)} \\ &\leq \frac{|y - z|^2 + 4(|x - y| + d_G(x))d_G(z)}{|y - z|^2 + 4|y - x|d_G(z)} \\ &= 1 + \frac{4d_G(x)d_G(z)}{|y - z|^2 + 4|y - x|d_G(z)} \leq 1 + \frac{4d_G(x)d_G(z)}{0 + 4td_G(x)d_G(z)} \\ &= 1 + \frac{1}{t}. \end{aligned}$$

Let us then assume $d_G(y) \neq d_{G'}(y)$ and $d_G(z) \neq d_{G'}(z)$. Now

$$\begin{aligned} \frac{p_{G'}^2(y, z)}{p_G^2(y, z)} &= \frac{|y - z|^2 + 4d_G(y)d_G(z)}{|y - z|^2 + 4d_{G'}(y)d_{G'}(z)} = \frac{|y - z|^2 + 4d_G(y)d_G(z)}{|y - z|^2 + 4|y - x||z - x|} \\ &\leq \frac{|y - z|^2 + 4(|x - y| + d_G(x))(|x - z| + d_G(x))}{|y - z|^2 + 4|y - x||z - x|} \\ &= 1 + \frac{4(|x - y|d_G(x) + |x - z|d_G(x) + d_G(x)^2)}{|y - z|^2 + 4|y - x||z - x|} \\ &\leq 1 + \frac{4(|x - y|d_G(x) + |x - z|d_G(x) + d_G(x)^2)}{4|y - x||z - x|} \\ &= 1 + \frac{|x - y|d_G(x)}{|y - x||z - x|} + \frac{|x - z|d_G(x)}{|y - x||z - x|} + \frac{d_G(x)^2}{|y - x||z - x|} \\ &\leq 1 + \frac{|x - y|d_G(x)}{|y - x|td_G(x)} + \frac{|x - z|d_G(x)}{td_G(x)|z - x|} + \frac{d_G(x)^2}{td_G(x)td_G(x)} \\ &= 1 + \frac{2}{t} + \frac{1}{t^2} = 1 + \frac{2t + 1}{t^2}. \end{aligned}$$

Combining the cases we obtain $c_p = \frac{t+1}{t}$.

Let us finally consider $m = s$. Now

$$\frac{s_{G'}(y, z)}{s_G(y, z)} = \frac{\inf_{u \in \partial G} |y - u| + |u - z|}{\inf_{u \in \partial G'} |y - u| + |u - z|}$$

and if the infimum in the denominator is obtained at a point $u \in \partial G$, then there is nothing to prove as $s_{G'}(y, z) = s_G(y, z)$ and we can choose $c_s = 1$. If this is not the case, then

$$\begin{aligned} \frac{s_{G'}(y, z)}{s_G(y, z)} &= \frac{\inf_{u \in \partial G} |y - u| + |u - z|}{\inf_{u \in \partial G'} |y - u| + |u - z|} = \frac{\inf_{u \in \partial G} |y - u| + |u - z|}{|y - x| + |x - z|} \\ &\leq \frac{|x - y| + d_G(x) + |x - z| + d_G(x)}{|y - x| + |x - z|} = 1 + \frac{2d_G(x)}{|y - x| + |x - z|} \\ &\leq 1 + \frac{2d_G(x)}{2td_G(x)} = 1 + \frac{1}{t} \end{aligned}$$

and we can choose $c_s = 1 + \frac{1}{t}$.

We see easily that $c_j, c_p, c_s \rightarrow 2$ as $t \rightarrow 1$. We show next that the constants c_j, c_p and c_s are best possible. In all three cases we consider $G = \mathbb{R}^n \setminus \{0\}$.

We start with $m = j$. Let $a > 0$. For points $x = e_1$, $y = (1+t)e_1$ and $z = (1+t+a)e_1$ we have by Lemma 3.36

$$\frac{j_{G'}(y, z)}{j_G(y, z)} = \frac{\log\left(1 + \frac{a}{t}\right)}{\log\left(1 + \frac{a}{1+t}\right)} \leq \lim_{a \rightarrow 0} \frac{\log\left(1 + \frac{a}{t}\right)}{\log\left(1 + \frac{a}{1+t}\right)} \leq \lim_{a \rightarrow 0} \frac{1+t+a}{t+a} = \frac{1+t}{t} \rightarrow 2$$

as $t \rightarrow 1$.

We next consider $m = p$. Let $a \in (0, t]$. For points $x = e_1$, $y = (1 + \sqrt{t^2 - a^2})e_1 + ae_2$ and $z = (1 + \sqrt{t^2 - a^2})e_1 - ae_2$ we have $|y - z| = 2a$ and

$$\frac{p_{G'}^2(y, z)}{p_G^2(y, z)} = \frac{|y - z|^2 + 4d_G(y)d_G(z)}{|y - z|^2 + 4d_{G'}(y)d_{G'}(z)} = \frac{4a^2 + 4\left(a^2 + \left(1 + \sqrt{t^2 - a^2}\right)^2\right)}{4a^2 + 4t^2}.$$

Now

$$\frac{p_{G'}^2(y, z)}{p_G^2(y, z)} \rightarrow \frac{4a^2 + 4\left(a^2 + \left(1 + \sqrt{1 - a^2}\right)^2\right)}{4a^2 + 4} = \frac{4a^2 + 8 + 8\sqrt{1 - a^2}}{4a^2 + 4}$$

as $t \rightarrow 1$ and

$$\frac{4a^2 + 8 + 8\sqrt{1 - a^2}}{4a^2 + 4} \rightarrow 4$$

as $a \rightarrow 0$.

We finally consider $m = s$. Let $a \in (0, t]$. For points $x = e_1$, $y = (1 + \sqrt{t^2 - a^2})e_1 + ae_2$ and $z = (1 + \sqrt{t^2 - a^2})e_1 - ae_2$ we have $|y - z| = 2a$ and

$$\begin{aligned} \frac{s_{G'}(y, z)}{s_G(y, z)} &= \frac{\frac{2a}{2t}}{\frac{2a}{2\sqrt{a^2 + (1 + \sqrt{t^2 - a^2})^2}}} = \frac{\sqrt{a^2 + \left(1 + \sqrt{t^2 - a^2}\right)^2}}{t} \\ &\rightarrow \sqrt{a^2 + \left(1 + \sqrt{1 - a^2}\right)^2} \end{aligned}$$

as $t \rightarrow 1$ and

$$\sqrt{a^2 + \left(1 + \sqrt{1 - a^2}\right)^2} = \sqrt{2 + 2\sqrt{1 - a^2}} \rightarrow 2$$

as $a \rightarrow 0$.

□

We show next that Theorem 3.39 does not work for the visual angle metric v . Let $G = \mathbb{R}^n \setminus \{0\}$ and $x = e_1$. Now for $y = \frac{e_1}{2}$ and $z = 2e_1$ we have $v_G(y, z) = 0$ and $v_{G \setminus \{x\}}(y, z) = \pi$.

Proof of Theorem 1.11 The result follows from Theorems 3.38 and 3.39.

□

4. SMOOTHNESS OF s -DISKS WITH SMALL RADII

In this section, we will consider the smoothness of triangular ratio metric balls in equilateral triangles and rectangles in \mathbb{R}^2 . Let $T_{\frac{\pi}{6},2}$ denote the equilateral triangle with vertex points $(0,0)$, $(\sqrt{3},1)$, $(\sqrt{3},-1)$, and $R_{a,b}$ denote the rectangle with vertex points (a,b) , $(a,-b)$, $(-a,b)$, $(-a,-b)$, where $a \geq b > 0$.

It is easy to see that the triangular ratio metric ball $B_{s_G}(x,r)$ is invariant under translations, stretchings, and orthogonal mappings. Hence, it is equivalent to consider the triangular ratio metric ball in the domain $T_{\frac{\pi}{6},2}$ and $R_{a,b}$.

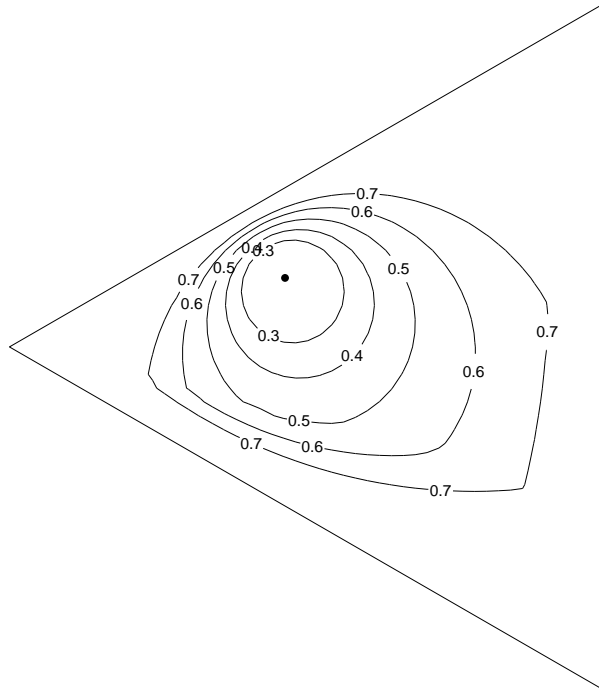


FIGURE 3. Triangular ratio metric balls $B_{s_G}(x, r)$ in $T_{\frac{\pi}{6},2}$.

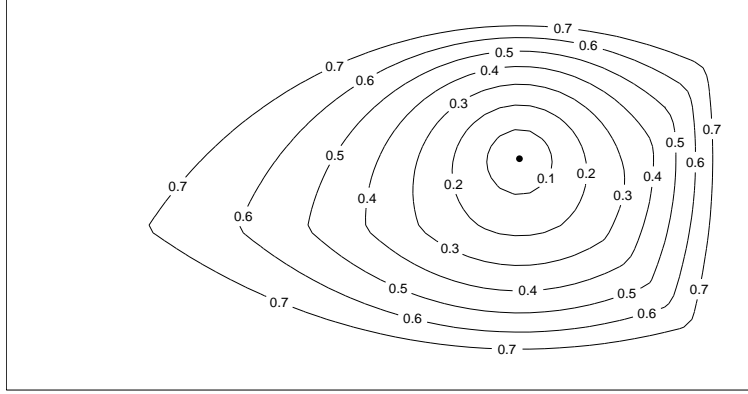
Lemma 4.1. *Let $P \subset \mathbb{R}^2$ be a polygon and suppose that there are half planes H_1, H_2, \dots, H_n such that*

$$P = \bigcap_{i=1}^n H_i.$$

Then for $x \in P$ and $r > 0$ we have

$$B_{s_P}(x, r) = \bigcap_{i=1}^n B_{s_{H_i}}(x, r).$$

Proof of Theorem 1.12 Denote by the lines $l_1 : y = \frac{\sqrt{3}}{3}x$, $l_2 : y = -\frac{\sqrt{3}}{3}x$, and $l_3 : x = \sqrt{3}$. For any point $x \in G = T_{\frac{\pi}{6},2}$ and $r \in (0,1)$, by lemma 3.2,

FIGURE 4. Triangular ratio metric balls $B_{s_G}(x, r)$ in $R_{a,b}$.

we have

$$B_{s_G}(x, r) = \cap_{i=1}^3 B_i,$$

where B_i is the corresponding triangular ratio metric ball $B_{s_{G_i}}(x, r)$, and G_i is the half plane with boundary line l_i . By elementary computation, we have that

$$B_1 : \left\{ y : \left(y_1 - \frac{(2-r^2)x_1 - \sqrt{3}r^2x_2}{2(1-r^2)} \right)^2 + \left(y_2 - \frac{(2+r^2)x_2 - \sqrt{3}r^2x_1}{2(1-r^2)} \right)^2 < \frac{r^2(x_1 - \sqrt{3}x_2)^2}{(1-r^2)^2} \right\},$$

$$B_2 : \left\{ y : \left(y_1 - \frac{(2-r^2)x_1 + \sqrt{3}r^2x_2}{2(1-r^2)} \right)^2 + \left(y_2 - \frac{(2+r^2)x_2 + \sqrt{3}r^2x_1}{2(1-r^2)} \right)^2 < \frac{r^2(x_1 + \sqrt{3}x_2)^2}{(1-r^2)^2} \right\},$$

and

$$B_3 : \left\{ y : \left(y_1 - \frac{x_1 - 2\sqrt{3}r^2 + x_1r^2}{1-r^2} \right)^2 + (y_2 - x_2)^2 < \frac{4r^2(x_1 - \sqrt{3})^2}{(1-r^2)^2} \right\}.$$

Hence, $B_{s_G}(x, r)$ is smooth if and only if $B_{s_G}(x, r)$ is one of the above three balls. By simply calculations, $B_1 \subset B_2$ and B_3 is equivalent to

$$0 < r \leq \frac{2x_2}{\sqrt{x_1^2 + x_2^2}}, \text{ and } 0 < r \leq \frac{x_2 - \sqrt{3}x_1 + 2}{\sqrt{(\sqrt{3} - x_1)^2 + (1 - x_2)^2}};$$

$B_2 \subset B_1$ and B_3 is equivalent to

$$0 < r \leq -\frac{2x_2}{\sqrt{x_1^2 + x_2^2}}, \text{ and } 0 < r \leq \frac{-x_2 - \sqrt{3}x_1 + 2}{\sqrt{(\sqrt{3} - x_1)^2 + (1 + x_2)^2}};$$

$B_3 \subset B_1$ and B_2 is equivalent to

$$0 < r \leq \frac{\sqrt{3}x_1 - x_2 - 2}{\sqrt{(\sqrt{3} - x_1)^2 + (1 - x_2)^2}} \quad \text{and} \quad 0 < r \leq \frac{\sqrt{3}x_1 + x_2 - 2}{\sqrt{(\sqrt{3} - x_1)^2 + (1 + x_2)^2}}.$$

That's for any point $x \in T_{\frac{\pi}{6}, 2}$, $0 < r < 1$, $B_{s_G}(x, r)$ is smooth if and only if

$$0 < r \leq \min \left\{ \frac{2|x_2|}{\sqrt{x_1^2 + x_2^2}}, \frac{|x_2| - \sqrt{3}x_1 + 2}{\sqrt{(x_1 - \sqrt{3})^2 + (1 - |x_2|)^2}} \right\},$$

or

$$0 < r \leq \frac{\sqrt{3}x_1 - 2 - |x_2|}{\sqrt{(x_1 - \sqrt{3})^2 + (1 - |x_2|)^2}}.$$

Obviously, for $x_2 = 0$ and $0 < x_1 \leq \frac{2\sqrt{3}}{3}$, or $|x_2| = \sqrt{3}x_1 - 2$, $B_{s_G}(x, r)$ cannot be smooth.

For the case $G = R_{a,b}$, let $l_1 : y = b$, $l_2 : x = a$, $l_3 : y = -b$, and $l_4 : x = -a$. For any point $x \in R_{a,b}$, and $r \in (0, 1)$, it follows from Lemma 3.2 that

$$B_{s_G}(x, r) = \cap_{i=1}^4 B_i,$$

where B_i is the corresponding triangular ratio metric ball $B_{s_{G_i}}(x, r)$, and G_i is the half plane with boundary line l_i . For any point $x \in R_{a,b}$, it follows from elementary computation that

$$B_1 : \left\{ y : (y_1 - x_1)^2 + \left(y_2 - \frac{x_2 + r^2x_2 - 2br^2}{1 - r^2} \right)^2 < \frac{4r^2(b - x_2)^2}{(1 - r^2)^2} \right\},$$

$$B_2 : \left\{ y : \left(y_1 - \frac{x_1 + r^2x_1 - 2ar^2}{1 - r^2} \right)^2 + (y_2 - x_2)^2 < \frac{4r^2(a - x_1)^2}{(1 - r^2)^2} \right\},$$

$$B_3 : \left\{ y : (y_1 - x_1)^2 + \left(y_2 - \frac{x_2 + r^2x_2 + 2br^2}{1 - r^2} \right)^2 < \frac{4r^2(b + x_2)^2}{(1 - r^2)^2} \right\},$$

and

$$B_4 : \left\{ y : \left(y_1 - \frac{x_1 + r^2x_1 + 2ar^2}{1 - r^2} \right)^2 + (y_2 - x_2)^2 < \frac{4r^2(a + x_1)^2}{(1 - r^2)^2} \right\}.$$

For $1 \leq i \leq 4$, let R_i denote the radius of B_i . If $x_2 > 0$, then $R_3 \geq R_1$. By calculations, $B_{s_G}(x, r) = B_1$ is equivalent to

$$0 < r \leq \min \left\{ \frac{x_2}{b}, \frac{(a - x_1) - (b - x_2)}{\sqrt{(a - x_1)^2 + (b - x_2)^2}}, \frac{(a + x_1) - (b - x_2)}{\sqrt{(a + x_1)^2 + (b - x_2)^2}} \right\}.$$

If $x_2 < 0$, then $R_1 \geq R_3$. By calculations, $B_{s_G}(x, r) = B_3$ is equivalent to

$$0 < r \leq \min \left\{ -\frac{x_2}{b}, \frac{(a - x_1) - (b + x_2)}{\sqrt{(a - x_1)^2 + (b + x_2)^2}}, \frac{(a + x_1) - (b + x_2)}{\sqrt{(a + x_1)^2 + (b + x_2)^2}} \right\}.$$

If $x_1 > 0$, then $R_4 \geq R_2$. By calculations, $B_{s_G}(x, r) = B_2$ is equivalent to

$$0 < r \leq \min \left\{ \frac{x_1}{a}, \frac{(b-x_2)-(a-x_1)}{\sqrt{(a-x_1)^2+(b-x_2)^2}}, \frac{(b+x_2)-(a-x_1)}{\sqrt{(a-x_1)^2+(b+x_2)^2}} \right\}.$$

If $x_1 < 0$, then $R_2 \geq R_4$. By calculations, $B_{s_G}(x, r) = B_4$ is equivalent to

$$0 < r \leq \min \left\{ -\frac{x_1}{a}, \frac{(b-x_2)-(a+x_1)}{\sqrt{(a+x_1)^2+(b-x_2)^2}}, \frac{(b+x_2)-(a+x_1)}{\sqrt{(a+x_1)^2+(b+x_2)^2}} \right\}.$$

That is, for any point $x \in R_{a,b}$, $0 < r < 1$, $B_{s_G}(x, r)$ is smooth if and only if

$$0 < r \leq \min \left\{ \frac{|x_2|}{b}, \frac{(a-|x_1|)-(b-|x_2|)}{\sqrt{(a-|x_1|)^2+(b-|x_2|)^2}} \right\},$$

or

$$0 < r \leq \min \left\{ \frac{|x_1|}{a}, \frac{(b-|x_2|)-(a-|x_1|)}{\sqrt{(a-|x_1|)^2+(b-|x_2|)^2}} \right\}.$$

Obviously, for $x_2 = 0$ and $a - |x_1| \geq b$, or $a - |x_1| = b - |x_2|$, $B_{s_G}(x, r)$ cannot be smooth. \square

5. QUASIREGULAR MAPS AND TRIANGULAR RATIO METRIC

In this section our goal is to summarize some basic facts about quasiconformal mappings, following closely [AVV], and [Vu1], and to prove Theorems 1.2 and 1.5. We assume that the reader is familiar with the basics of this theory. Here we adopt the standard definition of K -quasiconformality and K -quasiregularity from J. Väisälä's book [V] and from [Vu1], respectively. The first result is a quasiregular counterpart of the Schwarz lemma. Observe that the result is asymptotically sharp when $K \rightarrow 1$.

Theorem 5.1. [Vu1, Theorem 11.2, Lemma 7.22] *Let G be either \mathbb{B}^n or \mathbb{H}^n and $f : G \rightarrow fG \subset G$ a non-constant K -quasiregular mapping and let $\alpha = K_I(f)^{1/(1-n)}$. Then*

$$\begin{aligned} \tanh \left(\frac{1}{2} \rho_G(f(x), f(y)) \right) &\leq \varphi_K \left(\tanh \left(\frac{1}{2} \rho_G(x, y) \right) \right) \\ &\leq \lambda_n^{1-\alpha} \left(\tanh \left(\frac{1}{2} \rho_G(x, y) \right) \right)^\alpha, \end{aligned}$$

for all $x, y \in G$, where $\lambda_n \in [4, 2e^{n-1})$ is the Grötzsch ring constant depending only on n .

Proof of Theorem 1.2. (1) Because for all $x, y \in \mathbb{H}^n$,

$$s_{\mathbb{H}^n}(x, y) = \tanh \left(\frac{\rho_{\mathbb{H}^n}(x, y)}{2} \right),$$

by Theorem 5.1 the proof follows.

(2) By Theorems 5.1, 3.23 and Lemma 3.4 we have for all $x, y \in \mathbb{B}^n$,

$$\begin{aligned} s_{\mathbb{B}^n}(f(x), f(y)) &\leq \tanh\left(\frac{\rho_{\mathbb{B}^n}(f(x), f(y))}{2}\right) \\ &\leq \lambda_n^{1-\alpha} \tanh\left(\frac{\rho_{\mathbb{B}^n}(x, y)}{2}\right)^\alpha \\ &\leq \lambda_n^{1-\alpha} (2s_{\mathbb{B}^n}(x, y))^\alpha \\ &= 2^\alpha \lambda_n^{1-\alpha} (s_{\mathbb{B}^n}(x, y))^\alpha. \quad \square \end{aligned}$$

Corollary 5.2. *Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^n$ be a K -quasiregular mapping. Then for $x, y \in \mathbb{B}^n$ we have*

$$(5.3) \quad p_{\mathbb{B}^n}(f(x), f(y)) \leq 2^\alpha \lambda_n^{1-\alpha} (p_{\mathbb{B}^n}(x, y))^\alpha, \quad \alpha = K^{1/(1-n)}.$$

Proof. By Lemma 3.9, the proof is similar to the proof of Theorem 1.2. \square

By definition (1.1) it is clear that for $x, y \in G = \mathbb{R}^n \setminus \{0\}$, we have

$$s_G(x, y) = \frac{|x - y|}{|x| + |y|}.$$

Recall the following notation from [AVV, Section 14],

$$\eta_{K,n}^*(t) = \sup \{|g(x)| : |x| \leq t, g \in \mathcal{F}_K\},$$

$$\mathcal{F}_K = \{g : \mathbb{R}^n \rightarrow \mathbb{R}^n, g(0) = 0, g(e_1) = e_1, g \text{ is } K\text{-quasiconformal}\}.$$

Lemma 5.4. [AVV, 14.27] *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a K -quasiconformal mapping with $f(\infty) = \infty$, and let a, b, c be three distinct points in \mathbb{R}^n . Then*

$$\begin{aligned} \frac{1}{P_6(n, K)} \left(\frac{|a - c|}{|a - b| + |b - c|} \right)^\beta &\leq \frac{|f(a) - f(c)|}{|f(a) - f(b)| + |f(b) - f(c)|} \\ &\leq \frac{1}{P_5(n, K)} \left(\frac{|a - c|}{|a - b| + |b - c|} \right)^\alpha, \end{aligned}$$

where $\alpha = K^{1/(1-n)} = 1/\beta$ and $P_5(n, K) = 2^{1-(\beta/\alpha)} \lambda_n^{1-\beta} / \eta_{K,n}^*(1)$, $P_6(n, K) = 2^{1-(\alpha/\beta)} \lambda_n^{\beta-1} \eta_{K,n}^*(1)$. Here λ_n is as in Lemma 5.1 and $P_5(n, K) \rightarrow 1, P_6(n, K) \rightarrow 1$, when $K \rightarrow 1$.

Proof of Theorem 1.5. By Möbius invariance of the absolute ratio, the result follows from Lemma 5.4 if we take $b = f(b) = 0$. \square

Lemma 5.5. [AVV, 14.8] *For $n \geq 2$ and $K \geq 1$,*

$$\eta^*(1) \leq \exp(4K(K+1)\sqrt{K-1}).$$

Corollary 5.6. *Let $G = \mathbb{R}^n \setminus \{0\}$, and $f : G \rightarrow G$ be a K -quasiconformal mapping. If $n = 2$ then for $z, w \in G$,*

$$s_{fG}(f(z), f(w)) \leq \frac{\exp(4K(K+1)\sqrt{K-1}) - \pi(K-1)(1-\beta)}{2^{1-(\beta/\alpha)}} (s_G(z, w))^\alpha.$$

β and α are as in Lemma 5.4.

Proof. By [AVV, Corollary (10.33)], $\lambda(K) > \exp(\pi(K-1))$. Now by Lemma 5.5, the result follows immediately. \square

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